Homework I

Due Date: 26/02/2024

Exercise 1. Find the general solution to the following equations.

(i) (1 point)

$$x\partial_x u + y\partial_y u = xe^{-u}.$$

(ii) (1 point)

$$x(y^2+u)\partial_x u - y(x^2+u)\partial_y u = u(x^2-y^2).$$

Exercise 2. Consider the following Cauchy problems.

(i) (1 point) Solve

$$\begin{cases} \partial_x u + x\partial_y u = y, \quad (x,y) \in \mathbb{R}^2, \\ u(0,y) = \cos y, \quad y \in \mathbb{R}. \end{cases}$$

(ii) (1 point) Solve

$$\begin{cases} u\partial_x u + y\partial_y u = x, \quad (x,y) \in \mathbb{R}^2, \\ u(x,1) = 2x, \quad x \in \mathbb{R}. \end{cases}$$

Exercise 3. Consider the boundary value problem

$$\begin{cases} \partial_x u + x\partial_y u = 0, \quad (x,y) \in \mathbb{R}^2, \\ u(x,0) = f(x), \quad x \in \mathbb{R}. \end{cases}$$

(i) (1 point) Show that f is an even function to ensure the existence of a solution.

(ii) (1 point) Give the region of the xy-plane where the solution can be uniquely determined by the initial condition.

Exercise 4. For $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, we consider the Cauchy problem,

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = 0,$$

with

$$f(0,x,v) = f^0(x,v).$$

(i) (1 point) Using the method of characteristics prove that for $f^0 \in C^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, this problem has a unique solution $f \in C^1(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$.

(ii) (2 point) Prove that if in addition for $1 \leq p \leq \infty$, $f^0 \in L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ then for $t \in \mathbb{R}$, we have $f(t) \in L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, where $f(t)$ denotes the function $(x,v) \mapsto f(t,x,v)$, and

$$\|f(t)\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)} = \|f^0\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)}.$$

(iii) (1 point) Assume that $f^0 \in L^1(\mathbb{R}^3_x, L^\infty(\mathbb{R}^3_v))$. Prove that we have for $t \neq 0$,

$$\left| \int_{\mathbb{R}^3_v} f(t,x,v)\,dv \right| \leq \frac{1}{t^3} \|f^0\|_{L^1(\mathbb{R}^3_x, L^\infty(\mathbb{R}^3_v))}.$$